Problem 11307 (also problem 11292)(Proposed by David Callan, University of Wisconsin, Madison, WI)

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Show that if p is a prime and $p \geq 5$, then p^2 divides $\sum_{k=1}^{p^2-1}\binom{2k}{k}$.

Solution:

First note that the conclusion of the problem may be written as a congruence as follows:
$\sum_{k=1}^{p^2-1}\binom{2k}{k} \equiv 0 \pmod{p^2}$, or, equivalently as $\sum_{k=0}^{p^2-1}\binom{2k}{k} \equiv 1 \pmod{p^2}$.

In 2006 Pan ans Sum [1, Theorem 1.2] showed that

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod{p}. \tag{1}$$

where the Legendre symbol $\left(\frac{a}{3}\right) \in \{0, \pm 1\}$ satisfies $a \equiv \left(\frac{a}{3}\right) \pmod 3$

Note that equation (1), for $d = 0$ results

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p}. \tag{2}$$

This result is used by Sun and Tauraso [2] to prove amongst other congruencies, the following [2, Theorem 1.2, Equation (1.8)]

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d} \equiv \left(\frac{p^a-d}{3}\right) - p[p=3]\left(\frac{d}{3}\right) + 2p^a S_d \pmod{p^2} \tag{3}$$

where p is a prime, $a \in Z^+ = \{1, 2, 3, \ldots\}$, and $d \in \{0, 1, 2, 3, \ldots\}$. Also $[p = 3] = 1$ if $p = 3$ and $[p = 3] = 0$ in other case. S_d is a rational $p-$adic integer given by

$$S_d := \sum_{0<k<d}\frac{(-1)^{k-1}}{k}\left(\frac{d-k}{3}\right)$$

Note, that is our case, $S_d = 0$, p is a prime with $p \geq 5$, $d = 0$, and $a = 2$, and therefore Equation 3 gives

$$\sum_{k=0}^{p^2-1} \binom{2k}{k} \equiv \left(\frac{p^2}{3}\right) \pmod{p^2} \tag{4}$$

but, $\left(\frac{p^2}{3}\right) = 1$ and the problem is done.

References

[1] H. Pan and Z.W. Sun, A combinatorial identity with application to Catalan numbers, Discrete Math. **306** (2006), 1921–1940.

[2] Z.W. Sun and R. Tauraso, Congruences involving Catalan numbers, arXiv:0709.1665, (24/09/2007).