Problem 11333 (Proposed by Pablo Fernández Refolio, Universidad Autónoma de Madrid, Madrid, Spain)

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain. aplaza@dmat.ulpgc.es.

Show that
$$\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right)=\pi$$

Solution:

Fix an integer $n \geq 2$ and define for integer $i \geq 2$,
$$s(i)=\ln\left(\left(\frac{i^2-1}{i^2}\right)^{2(i^2-1)}\left(\frac{i+1}{i-1}\right)^{i}\right)$$

If $S(n)=\sum_{i=2}^{n+1} s(i)$, then $\lim_{n\to\infty} S(n)=\ln\prod_{n=2}^{\infty}\left(\left(\frac{n^2-1}{n^2}\right)^{2(n^2-1)}\left(\frac{n+1}{n-1}\right)^{n}\right)$, and so the proposed product is equal to e^S, where $S=\lim_{n\to\infty} S(n)$.

$$\begin{aligned} s(i) &= 2(i^2-1)\ln\left(\frac{i^2-1}{i^2}\right)+i\ln(i+1)-i\ln(i-1)\\ &= \big(2(i^2-1)-i\big)\ln(i-1)-4(i^2-1)\ln i+\big(2(i^2-1)+i\big)\ln(i+1)\end{aligned}$$

$$S(n)=\sum_{i=2}^{n+1} s(i)=\sum_{i=2}^{n+2} c(i)\ln(i),$$

with $c(i)$ polynomial functions in i.

We claim
$$c(i)=\left\{\begin{array}{ll} 1 & \text{if } i=2,\\ 2 & \text{for } 3\leq i\leq n,\\ -2n^2-7n-2 & \text{if } i=n+1,\\ 2n^2+5n+1 & \text{if } i=n+2,\\ 0 & \text{for } i>n+2. \end{array}\right\} \tag{1}$$

The proof of (1) is straightforward. For example for $3 \le i \le n$ the contribution to $c(i)$ from $s(i-1)$, $s(i)$, and $s(i+1)$ respectively, is $2\left((i-1)^2-1\right)+i-1$, $-4(i^2-1)$, and $2\left((i+1)^2-1\right)-(i+1)$ which sums to 2 as required. Proofs of the other cases of (1) are treated similarly.

It follows from (1) that

$$S(n) = \ln\left(2\left(\frac{n!}{2}\right)^2 (n+1)^{-2n^2-7n-2}(n+2)^{2n^2+5n+1}\right) \tag{2}$$

$$S(n) = \ln\left(\frac{(n!)^2}{2}\left(\frac{n+2}{n+1}\right)^{2n^2+5n+1}\left(\frac{1}{n+1}\right)^{2n+1}\right) \tag{3}$$

To evaluate (3) as $n \to \infty$ we use the following formulae:

$$\left\{\begin{array}{ll} (n!)^2 \sim (2\pi n)(\frac{n}{e})^{2n} & \text{Stirling's formula,} \\ \left(\frac{n+2}{n+1}\right)^{2n^2+5n+1} \sim e^{2n+2} & \\ \left(\frac{1}{n+1}\right)^{2n+1} \sim n^{-2n-1}e^{-2} & \end{array}\right\} \tag{4}$$

The second equation in (4) follows by taking logarithms and using Taylor's formula:

$$\lim_{n\to\infty} \ln\left(e^{-2n}\left(\frac{n+2}{n+1}\right)^{2n^2+5n+1}\right) =$$
$$= \lim_{n\to\infty}\left(-2n + \frac{2n^2+5n+1}{n+1} - \frac{1}{2}\frac{2n^2+5n+1}{(n+1)^2} + o(1)\right) = 2$$

proving the second formula in (3).

The third equation in (4) follows from the fact:

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n+1} = e^{-2}$$

Substituting the limits of (4) into (2) and performing some straightforward cancellations shows

$$S = \lim_{n\to\infty} S(n) = \ln \pi$$

from where the proposed product is obtained.