Solution to Problem # B-1044

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Problem# B-1044 *Proposed by Paul S. Bruckman, Sointula, Canada*
Prove the following identities:

(1) $(L_n)^2 = 2(F_{n+1})^2 - (F_n)^2 + 2(F_{n-1})^2$:
(2) $25(F_n)^2 = 2(L_{n+1})^2 - (L_n)^2 + 2(L_{n-1})^2$.

Solution. We use the following relations between Fibonacci and Lucas numbers: $L_n = F_{n+1} + F_{n-1}$, and $5F_n = L_{n+1} + L_{n-1}$, which follow directly from the Binet formulas.

Proof of (1): Since $L_n = F_{n+1} + F_{n-1}$, we have:

$$\begin{aligned}
(L_n)^2 &= (F_{n+1})^2 + (F_{n-1})^2 + F_{n+1}F_{n-1} + F_{n+1}F_{n-1} \\
&= (F_{n+1})^2 + (F_{n-1})^2 + F_{n+1}(F_{n+1} - F_n) + (F_n + F_{n-1})F_{n-1} \\
&= 2(F_{n+1})^2 + 2(F_{n-1})^2 - F_n(F_{n+1} - F_{n-1}) \\
&= 2(F_{n+1})^2 + 2(F_{n-1})^2 - (F_n)^2
\end{aligned}$$

□

Proof of (2): Since $5F_n = L_{n+1} + L_{n-1}$, we have:

$$\begin{aligned}
25(F_n)^2 &= (L_{n+1})^2 + (L_{n-1})^2 + L_{n+1}L_{n-1} + L_{n+1}L_{n-1} \\
&= (L_{n+1})^2 + (L_{n-1})^2 + L_{n+1}(L_{n+1} - L_n) + (L_n + L_{n-1})L_{n-1} \\
&= 2(L_{n+1})^2 + 2(L_{n-1})^2 - L_n(L_{n+1} - L_{n-1}) \\
&= 2(L_{n+1})^2 + 2(L_{n-1})^2 - (L_n)^2
\end{aligned}$$

□

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