Solution to Problem # B-1045

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Problem# B-1045 *Proposed by H.-J. Seiffert, Berlin, Germany*
Show that, for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} \frac{F_k}{F_{k+1}} \left(\prod_{j=k}^{n} \frac{F_j}{L_j} \right)^2 = \frac{F_n}{F_{n+1}}$$

Solution. By induction.
For $n = 1$ the equality becomes trivial. Let us suppose the equality is true for integer n. Then for $n + 1$ we have to prove that

$$\sum_{k=1}^{n+1} 4^{n+1-k} \frac{F_k}{F_{k+1}} \left(\prod_{j=k}^{n+1} \frac{F_j}{L_j} \right)^2 = \frac{F_{n+1}}{F_{n+2}}$$

We denote by $(LHS)_{n+1}$ the Left-Hand Side (LHS) of the previous equality, that is for the case $n + 1$, and respectively by $(LHS)_n$ for n. Then we have:

$$(LHS)_{n+1} = \left(\frac{F_{n+1}}{L_{n+1}} \right)^2 \left[4(LHS)_n + \frac{F_{n+1}}{F_{n+2}} \right]$$

Where, by the induction hypothesis:

$$(LHS)_{n+1} = \left(\frac{F_{n+1}}{L_{n+1}} \right)^2 \left[4\frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}} \right]$$

$$\begin{aligned} 4\frac{F_n}{F_{n+1}} + \frac{F_{n+1}}{F_{n+2}} &= \frac{4F_nF_{n+2} + F_{n+1}^2}{F_{n+1}F_{n+2}} = \frac{4F_nF_{n+1} + 4F_nF_n + F_{n+1}^2}{F_{n+1}F_{n+2}} \\ &= \frac{(F_{n+1} + 2F_n)^2}{F_{n+1}F_{n+2}} = \frac{(F_{n+2} + F_n)^2}{F_{n+1}F_{n+2}} \\ &= \frac{(L_{n+1})^2}{F_{n+1}F_{n+2}} \end{aligned}$$

And therefore,

$$(LHS)_{n+1} = \left(\frac{F_{n+1}}{L_{n+1}} \right)^2 \frac{(L_{n+1})^2}{F_{n+1}F_{n+2}} = \frac{F_{n+1}}{F_{n+2}}.$$

Which proves the equality for the case $n + 1$. $\square$

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