893. Proposed by Ovidiu Furdui, University of Toledo, Toledo, Ohio.

Let f be any function that has a Taylor series representation at 0 with radius of convergence 1, and let

$$T_n(x) = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

denote the nth-degree Taylor polynomial of f. Find the sum

$$\sum_{n=1}^{\infty} x^n \left(f(x) - T_n(x)\right)$$

for $|x| < 1$.

Solution: (by Ángel Plaza, Sergio Falcón, and José M. Pacheco, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)

Note that the proposed expression may be written as follows:

$$\sum_{n=1}^{\infty} x^n \left(f(x) - T_n(x)\right) = \sum_{n=0}^{\infty} x^n \left(f(x) - T_n(x)\right) - \left(f(x) - f(0)\right)$$

In our case since function $f(x)$ has a Taylor series representation, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$. Also $f(x^2) = \sum_{n=0}^{\infty} a_n x^n x^n$.

Now by a property of the ordinary power series generating functions[1]: $\dfrac{f(x)}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n$, and $\dfrac{f(x^2)}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k x^k \right) x^n$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} x^n \left(f(x) - T_n(x)\right) &= f(x) \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} T_n(x) x^n \\ &= \frac{f(x)}{1-x} - \frac{f(x^2)}{1-x} \end{aligned}$$

From which the proposed sum results:

$$\sum_{n=1}^{\infty} x^n \left(f(x) - T_n(x)\right) = \frac{f(x)}{1-x} - \frac{f(x^2)}{1-x} - \left(f(x) - f(0)\right) \qquad \square$$

[1] *H. Wilf, Generatingfunctionology, Academic Press, Second Edition, New York, 1994.*