

906. Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.

(a) Find the value of

$$\prod_{n=1}^{\infty} \left(\frac{2m+2n-1}{2n-1} \right) \left(\frac{2n}{2m+2n} \right)$$

(b) More generally, if the real part of z is positive, find the value of:

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)^{(-1)^{n-1}}$$

Solution: BY SANTIAGO JOSÉ DE LUXÁN (STUDENT) AND ANGEL PLAZA,
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$$\begin{aligned} (a) P &= \prod_{n=1}^{\infty} \left(\frac{2m+2n-1}{2n-1} \right) \left(\frac{2n}{2m+2n} \right) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \left(\frac{2m+2n-1}{2n-1} \right) \left(\frac{2n}{2m+2n} \right) \\ P_N &= \prod_{n=1}^N \left(\frac{2m+2n-1}{2n-1} \right) \left(\frac{2n}{2m+2n} \right) = \frac{(2N+2m-1)!!}{(2m-1)!!(2N-1)!!} \frac{N!m!}{(N+m)!} \\ &= \frac{m!}{(2m-1)!!} \frac{(2N+2m-1)!!}{(2N-1)!!} \frac{N!}{(N+m)!} \end{aligned}$$

It is known that for odd integer values of n we have:

$$\Gamma\left(\frac{n}{2} + 1\right) = \sqrt{\pi} \frac{n!!}{2^{(n+1)/2}} \Rightarrow n!! = \frac{\Gamma\left(\frac{n}{2} + 1\right) 2^{(n+1)/2}}{\sqrt{\pi}}$$

$$\begin{aligned} (2N+2m-1)!! &= \frac{\Gamma(N+m+\frac{1}{2}) 2^{N+m}}{\sqrt{\pi}} \\ (2N-1)!! &= \frac{\Gamma(N+\frac{1}{2}) 2^N}{\sqrt{\pi}} \end{aligned}$$

and therefore,

$$P_N = \frac{m! 2^m}{(2m-1)!!} \cdot \frac{\Gamma(N+m+\frac{1}{2})}{\Gamma(N+\frac{1}{2})} \cdot \frac{\Gamma(N+1)}{\Gamma(N+m+1)}$$

Now since, $\frac{\Gamma(z+m)}{\Gamma(z)} = (z)_m$, then $\lim_{N \rightarrow \infty} \frac{\Gamma(N+m+\frac{1}{2})}{\Gamma(N+\frac{1}{2})} \cdot \frac{\Gamma(N+1)}{\Gamma(N+m+1)} = 1$,
and $P = \frac{(2m)!!}{(2m-1)!!}$. \square

$$(b) P(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{(-1)^{n-1}} = \prod_{n=1}^{\infty} \frac{1 + \frac{z}{2n-1}}{1 + \frac{z}{2n}} = (z+1) \prod_{n=1}^{\infty} \frac{1 + \frac{z}{2n+1}}{1 + \frac{z}{2n}}$$

We use the product expression of the gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + \frac{z}{n}}, \text{ and so } \Gamma\left(\frac{z}{2}\right) = 2 \frac{e^{-\gamma z/2}}{z} \prod_{n=1}^{\infty} \frac{e^{z/2n}}{1 + \frac{z}{2n}}$$

from where

$$\prod_{n=1}^{\infty} \frac{1}{1 + \frac{z}{2n}} = \frac{z}{2} \Gamma\left(\frac{z}{2}\right) e^{\gamma z/2} \prod_{n=1}^{\infty} e^{-z/2n}$$

On the other hand,

$$\begin{aligned} 1 + \frac{z}{2n+1} &= \frac{2n}{2n+1} \left(1 + \frac{z}{2n}\right) = \frac{1}{1 + \frac{1}{2n}} \left(1 + \frac{z}{2n}\right) \\ \prod_{n=1}^{\infty} \left(1 + \frac{z}{2n+1}\right) &= \prod_{n=1}^{\infty} \frac{1}{1 + \frac{1}{2n}} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z+1}{2n}\right) \\ &= \prod_{n=1}^{\infty} \frac{1}{1 + \frac{1}{2n}} \cdot \frac{2e^{-\gamma \frac{z+1}{2}}}{(z+1)\Gamma(z+1)} \prod_{n=1}^{\infty} e^{\frac{z+1}{2n}} \end{aligned}$$

Therefore,

$$P(z) = \frac{e^{-\gamma/2}}{1/2} \prod_{n=1}^{\infty} \frac{e^{1/2n}}{1 + \frac{1}{2n}} \frac{\frac{z}{2} \Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{z+1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{z}{2} + 1\right)}{\Gamma\left(\frac{z+1}{2}\right)}$$