907. Proposed by Brian Bradie, Christopher Newport University, Newport News, Virginia.

For each nonnegative integer n, let $a_n = \left(\sum_{j=0}^{n} Q_j\right)^2 - \sum_{j=0}^{n} Q_{2j+1}$, where Q_n is the nth Pell-Lucas number; that is $Q_0 = 2$, $Q_1 = 2$, and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Evaluate

$$\sum_{j=0}^{\infty} \frac{a_n}{n!}$$

Solution: by Sergio Falcon and Angel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain

By solving the recurrence relation of the Pell–Lucas numbers, $r^2 = 2r + 1$, we obtain the Binet formula $Q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$.

Then, by the partial sum of a geometric series we get

$$\begin{aligned}
\sum_{j=0}^{n} Q_j &= \sum_{j=0}^{n} \left(1+\sqrt{2}\right)^j + \left(1-\sqrt{2}\right)^j \\
&= \frac{1-\left(1+\sqrt{2}\right)^{n+1}}{1-1-\sqrt{2}} + \frac{1-\left(1-\sqrt{2}\right)^{n+1}}{1-1+\sqrt{2}} \\
&= \frac{\left(1+\sqrt{2}\right)^{n+1} - \left(1-\sqrt{2}\right)^{n+1}}{\sqrt{2}} \\
\left(\sum_{j=0}^{n} Q_j\right)^2 &= \frac{\left(1+\sqrt{2}\right)^{2n+2} - \left(1-\sqrt{2}\right)^{2n+2}}{2} + (-1)^n \\
\sum_{j=0}^{n} Q_{2j+1} &= \frac{1+\sqrt{2}-\left(1+\sqrt{2}\right)^{2n+3}}{1-(1+\sqrt{2})^2} + \frac{1-\sqrt{2}-\left(1-\sqrt{2}\right)^{2n+3}}{1-(1-\sqrt{2})^2} \\
&= \frac{1+\sqrt{2}-\left(1+\sqrt{2}\right)^{2n+3}}{-2-2\sqrt{2}} + \frac{1-\sqrt{2}-\left(1-\sqrt{2}\right)^{2n+3}}{-2+2\sqrt{2}} \\
&= \frac{\left(1+\sqrt{2}\right)^{2n+2} - 1}{2} + \frac{\left(1-\sqrt{2}\right)^{2n+2} - 1}{2}
\end{aligned}$$

From where $a_n = (-1)^n + 1$, and therefore

$$\sum_{j=0}^{\infty} \frac{a_n}{n!} = \sum_{j=0}^{\infty} \frac{(-1)^n + 1}{n!} = e^{-1} + e$$

□