923. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.*

Let $\alpha \in (0,1)$ be an arbitrary irrational number. Construct a sequence $\{a_n\}$ of real numbers with the following two properties:

(a) $\{a_n \mid n \in \mathbb{N}\} \subset \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{\frac{n-1}{n} \mid n \in \mathbb{N}\}$,

(b) $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = \alpha$.

Solution: (by Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)

The sequence $\{a_n\}$ is constructed recursively as follows:

If $\alpha \in (0,1/2)$, a_1 is the first number in the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$, say $\frac{1}{m_1}$ such that $0 < a_1 = \frac{1}{m_1} < \alpha < \frac{1}{m_1-1} \leq 1/2$.

Otherwise, if $\alpha \in (1/2,1)$ a_1 is the first number in the set $\{\frac{n-1}{n} \mid n \in \mathbb{N}\}$, say $\frac{m_1-1}{m_1}$ such that $1/2 \leq < \frac{m_1-2}{m_1-1} < \alpha < a_1 = \frac{m_1-1}{m_1} < 1$.

It is clear that a_1 so defined verifies the first condition required and besides $|\alpha - a_1| < \frac{1}{m_1} \leq 1/2 < 1$. In addition, it follows straightforwardly that $2\alpha - a_1$ belongs to the interval $(0,1)$.

Let us suppose now that we have already constructed $a_1, a_2, \ldots, a_{n-1}$. Let $\alpha_{n-1} = n\alpha - \sum_{i=1}^{n-1} a_i$. The next term of the sequence we are looking for, a_n is defined as follows:

If $\alpha_{n-1} \in (0,1/2)$, then a_n is the first number in the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$, say $\frac{1}{m_n}$ such that $0 < a_n = \frac{1}{m_n} < \alpha_{n-1} < \frac{1}{m_n-1} \leq 1/2$.

Otherwise, if $\alpha_{n-1} \in (1/2,1)$ a_n is first number in the set $\{\frac{n-1}{n} \mid n \in \mathbb{N}\}$ say $\frac{m_n-1}{m_n}$ such that $1/2 \leq \frac{m_n-2}{m_n-1} < \alpha < a_1 = \frac{m_n-1}{m_n} < 1$.

Note that in both cases $|\alpha_{n-1} - a_n| < \frac{1}{m_n} < 1$.

Let us prove now that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = \alpha$:

$$\left|\alpha - \frac{a_1+a_2+\cdots+a_n}{n}\right| = \left|\frac{n\alpha - \sum_{i=1}^{n-1} a_i - a_n}{n}\right| < \frac{1/m_n}{n} < \frac{1}{n},$$

which implies that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots a_n}{n} = \alpha$. $\square$