Solution to CMJ Problem # 925

Ángel Plaza and Sergio Falcón
Department of Mathematics, Universidad de Las Palmas de Gran Canaria
35017–Las Palmas G.C. SPAIN, aplaza@dmat.ulpgc.es

925. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania and Tudorel Lupu (student), Decebal High School, Constanta, Romania.*

Let f a twice differentiable function on $\mathbb{R}$ with f'' continuous on $[0,1]$ such that

$$\int_0^1 f(x)dx = 2\int_{1/4}^{3/4} f(x)dx$$

Prove that there exists an $x_0 \in (0,1)$ such that $f''(x_0) = 0$.

Solution: Bellow there are two proofs:

(1) Letting $g(x) = f(1/2 + x)$ and

$$G(t) = \int_{-t}^{t} g(x)dx - 2\int_{-t/2}^{t/2} g(x)dx,$$

then $G(0) = 0$ and the required condition is equivalent to $G(1/2) = 0$. Hence, by the Mean Value Theorem there is $t_0 \in (0, 1/2)$ such that $G'(t_0) = 0$. Since

$$G'(t) = g(t) - g(t/2) - (g(-t/2) - g(-t))$$

then by the Mean Value Theorem, there are $\theta_+ \in (t_0/2, t_0)$ and $\theta_- \in (-t_0, -t_0/2)$ such that

$$0 = G'(t_0) = g'(\theta_+)t_0/2 - g'(\theta_-)t_0/2.$$

By applying again the Mean Value Theorem there is $\theta \in (\theta_-, \theta_+) \subset (-1/2, 1/2)$ such that

$$0 = g''(\theta)(\theta_+ - \theta_-)$$

that is, letting $x_0 = 1/2 + \theta \in (0,1)$, $f''(x_0) = g''(\theta) = 0$. $\square$

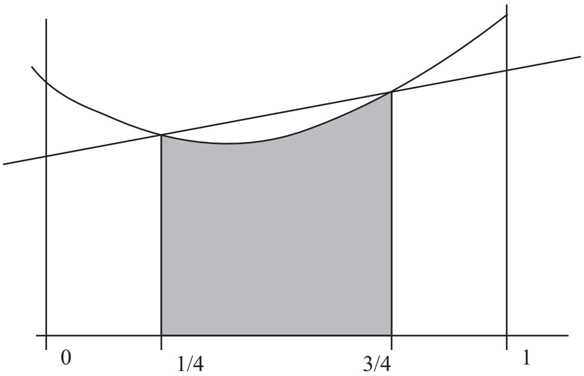
Note that the problem may be slightly generalized as follows: Given $a, b \in (0,1)$ with $a < b$, if f a twice differentiable function on $\mathbb{R}$ with f'' continuous on $[0,1]$ such that

$$\int_0^1 f(x)dx = \frac{\int_a^b f(x)dx}{b-a}$$

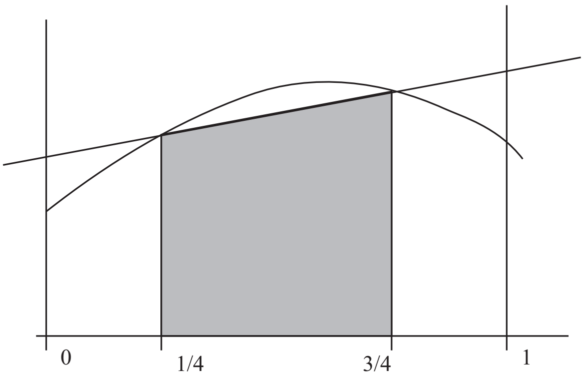
then there is some $x_0 \in (0,1)$ such that $f''(x_0) = 0$.

(2) The problem may be proved by contradiction and graphically as follows:

(a) If $f''(x) > 0$ for all $x \in (0,1)$. Then $\displaystyle\int_0^1 f(x)dx > \frac{\int_{1/4}^{3/4} f(x)dx}{1/2}$:



(b) If $f''(x) < 0$ for all $x \in (0,1)$. Then $\displaystyle\int_0^1 f(x)dx < \frac{\int_{1/4}^{3/4} f(x)dx}{1/2}$:



Remark: It is also true that if f a twice differentiable function on $\mathbb{R}$ with f'' continuous on $[0,1]$ and there is a not empty subinterval $(a,b) \subset [0,1]$ such that $\displaystyle\int_0^1 f(x)dx = \frac{\int_a^b f(x)dx}{b-a}$ there exists an $x_0 \in (0,1)$ such that $f''(x_0) = 0$. $\square$