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Consider the polynomial $f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$, where a, b, c, and d are positive real numbers. Prove that if f has four positive roots, then $a > b > c > d$.

Solution: *(by Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)*

Let us called $f_4(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$. Then $f_4'(x) = 4(x^3 - 3ax^2 + 3b^2x - c^3)$, so $f_4''(x) = 4 \cdot 3(x^2 - 2ax + b^2)$.

Let us denote $f_3(x) = x^3 - 3ax^2 + 3b^2x - c^3$ and $f_2(x) = x^2 - 2ax + b^2$. Note that $f_3(x)$ and $f_2(x)$ have the same roots as $f_4'(x)$ and $f_4''(x)$ respectively.

By hypothesis, $f_4(x)$ has four positive roots. Rolle's Theorem guarantees that $f_3(x)$ has three positive roots, and $f_2(x)$ has two positive roots. Moreover, if we denote by $r_{i,j}$ the j-th positive root of $f_i(x)$, then $r_{i,j} < r_{i-1,j} < r_{i,j+1}$. See Figure 1 in which a possible situation of the roots of these polynomials is presented.

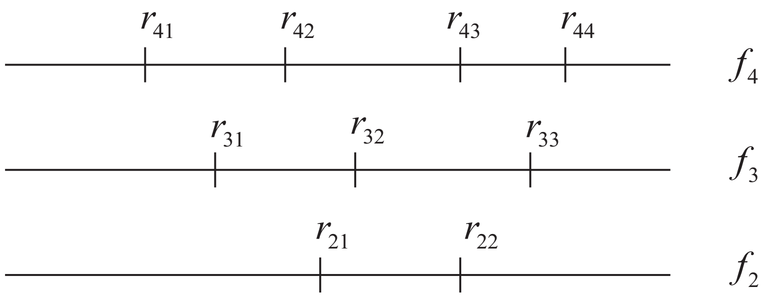

Figure 1: Roots of polynomials f_4, f_3, and f_2

We know that $f_2(x) = x^2 - 2ax + b^2$ has two positive roots, and hence $4a^2 - 4b^2 > 0$. Since a, b are positive, then $a > b$. The roots of f_2 are $r_{21} = a - \sqrt{a^2 - b^2}$ and $r_{21} = a + \sqrt{a^2 - b^2} > a$, and $r_{21} < b < a < r_{21}$.

Suppose now, that $c \geq b > 0$, then $f_3(x) = x^3 - 3ax^2 + 3b^2x - c^3 \leq x^3 - 3bx^2 + 3b^2x - b^3 = (x-b)^3$. And therefore, the roots of f_3, $r_{3,j}$ verify $0 < r_{3,j} < b$. But this statement is a contradiction with the fact that $b < a$, and at least one of the roots of f_3 is greater than $r_{21} = a + \sqrt{a^2 - b^2} > a$. See Figure 1. Therefore, $b > c$.

It remais to prove that $c > d$. Let us suppose that $d \geq c$. Since $a > b > c$, then

$$\begin{aligned} f_4(x) &= x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4 \\ &> x^4 - 4ax^3 + 6c^2x^2 - 4c^3x + c^4 \\ &= (x-c)^4 + 4(a-c)x^3 \end{aligned}$$

But since $a \geq c$, then $(x-c)^4 + 4(a-c)x^3 \geq 0$ for all positive real x, so $f_4(x) > 0$ for all positive real x, and we have found a contradiction and the proof is complete. $\square$

Note that the same argument applies to the general case:

Consider the polynomial

$$f_n(x) = x^n - \binom{n}{1} a_1 x^{n-1} + \cdots + (-1)^{n-1} \binom{n}{n-1} a_{n-1}^{n-1} x + (-1)^n a_n^n,$$

where $a_1, a_2, \ldots, a_n$ are positive real numbers. If f has n positive roots, then $a_1 > a_2 > \ldots > a_n$.