

**887.** Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.

Evaluate

$$\sum_{k=1}^{2n+1} \frac{2 - \overline{z_k}}{2 + z_k},$$

where  $z_1, z_2, \dots, z_{2n+1}$  are the  $(2n+1)$ 'st roots of unity, and  $\overline{z_k}$  denotes the complex conjugate of  $z$ .

**Solution:** (by Ángel Plaza, Sergio Falcón, and José M. Pacheco, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)

First, notice that for any root of unity  $z_k$ ,  $\overline{z_k} = z_k^{-1}$  so the general term in the sum can be written as:

$$\frac{2 - \overline{z_k}}{2 + z_k} = \frac{2 - z_k^{-1}}{2 + z_k} = \frac{2z_k - 1}{z_k(2 + z_k)} = \frac{-1/2}{z_k} + \frac{5/2}{2 + z_k}$$

Now, since  $z_k^{-1}$  is also a root of unity, then  $\sum_{k=1}^{2n+1} \frac{-1/2}{z_k} = 0$  and the proposed sum is reduced to

$$S_n = \sum_{k=1}^{2n+1} \frac{2 - \overline{z_k}}{2 + z_k} = \sum_{k=1}^{2n+1} \frac{5/2}{2 + z_k}$$

Let  $z_k = e^{\alpha_k} i$ , where  $\alpha_k = \frac{2(k-1)\pi}{2n+1}$ . Then  $z_1 = 1$ , and for every  $k \in \{1, \dots, n\}$ ,  $z_{2n-k+1} = \overline{z_k}$ . Therefore the corresponding terms in the sum may be grouped:

$$\frac{1}{2 + z_k} + \frac{1}{2 + \overline{z_k}} = \frac{4 + 2 \cos \alpha_k}{5 + 4 \cos \alpha_k} = \frac{1}{2} + \frac{3/2}{5 + 4 \cos \alpha_k}$$

$$S_n = \sum_{k=1}^{2n+1} \frac{5/2}{2 + z_k} = \frac{5}{2} \left( \frac{1}{3} + \frac{n}{2} + \frac{3}{2} \sum_{k=1}^n \frac{1}{5 + 4 \cos \alpha_k} \right)$$

Now we use the following theorem (D. Andrica & M. Piticari, On some interesting trigonometric sums, *Acta universitatis apulensis*, 15 (2008) 299-308):

*For any real number  $a \in R - \{-1, 1\}$  the following relation holds:*

$$\sum_{k=0}^{n-1} \frac{1}{a^2 - 2a \cos \frac{2k\pi}{n} + 1} = \frac{n(a^n + 1)}{(a^2 - 1)(a^n - 1)}$$

Applying it to our problem, we readily obtain:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{5 + 4 \cos \alpha_k} &= \frac{1}{2} \left( \sum_{k=1}^{2n+1} \frac{1}{2^2 + 2 \cdot 2 \cos \alpha_k + 1} - \frac{1}{9} \right) = \\ &= \frac{1}{2} \left( \frac{(2n+1)(1 - 2^{2n+1})}{3(2^{2n+1} + 1)} - \frac{1}{9} \right) \\ S_n &= \frac{5}{2} \left( \frac{1}{3} + \frac{n}{2} + \frac{3}{2} \cdot \frac{1}{2} \left( \frac{(2n+1)(2^{2n+1} - 1)}{3(2^{2n+1} + 1)} - \frac{1}{9} \right) \right) = \\ &= \frac{5(2n+1)}{4} \left( 1 - \frac{1}{2^{2n+1} + 1} \right) \end{aligned}$$

□