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Evaluate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\sqrt{n/k^3}}.$$

Solution:

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Let us call $L_n=\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\sqrt{n/k^3}}$ and $L=\lim_{n\to\infty}L_n$.

Then, by taking logarithms, we have:

$$\ln L_n=\sum_{k=1}^{n}\sqrt{\frac{n}{k^3}}\ln\left(1+\frac{k}{n}\right)=\sum_{k=1}^{n}\frac{\sqrt{\left(\frac{n}{k}\right)^3}\ln\left(1+\frac{k}{n}\right)}{n}$$

$$\ln L=\lim_{n\to\infty}\ln L_n=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{\left(\frac{n}{k}\right)^3}\ln\left(1+\frac{k}{n}\right)}{n}=\int_0^1\frac{\ln(1+x)}{x^{3/2}}dx=I$$

For solving the improper integral I we change $x=t^2$, $dx=2t\ dt$ and then

$$I=\int_0^1\frac{\ln(1+t^2)}{t^3}2tdt=2\int_0^1\frac{\ln(1+t^2)}{t^2}dt$$

Now, integrating by parts, with $u=\ln(1+t^2)$ and $dv=\frac{dt}{t^2}$ we obtain:

$$I=\left.\frac{-2\ln(1+t^2)}{t}\right]_0^1+2\int_0^1\frac{2}{1+t^2}dt=-2\ln 2+4\arctan 1=\pi-2\ln 2.$$

(Note that $\lim_{t\to 0^+}\frac{\ln(1+t^2)}{t}=0$).
Therefore, $L=e^{\pi-\ln 4}=\frac{e^{\pi}}{4}$. $\square$