Problem B-951

B-951 ***Proposed by the Stanley Rabinowicz, MathPro Press, Westford, MA (February 2003)***

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

Solution by Angel Plaza and Sergio Falcón (jointly), Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

The first values for Fibonacci numbers, Lucas numbers, and sequence u_n lead us to conjecture that

$$u_n = \frac{F_{2n-1}}{L_{2n-1}} \tag{1}$$

Proof: (By induction on n) For $n = 1$, $u_1 = 1$ and $\dfrac{F_{2-1}}{L_{2-1}} = \dfrac{F_1}{L_1} = 1$.

Suppose that (1) is true for u_n, and let us prove it for u_{n+1}:

By definition, $u_{n+1} = \dfrac{3u_n + 1}{5u_n + 3}$. So, taking into account (1) we get:

$u_{n+1} = \dfrac{3u_n + 1}{5u_n + 3} = \dfrac{3F_{2n-1} + L_{2n-1}}{5F_{2n-1} + 3L_{2n-1}}$

Now, from [1]:

$$\left\{ \begin{array}{ccc} F_n + F_{n+2} & = & L_{n+1} \\ L_n + L_{n+2} & = & 5F_{n+1} \end{array} \right\}$$

from which after some algebra we get: $3F_{2n-1} + L_{2n-1} = 2F_{2n+1}$, and $5F_{2n-1} + 3L_{2n-1} = 2L_{2n+1}$, and hence $u_{n+1} = \dfrac{2F_{2n+1}}{2L_{2n+1}} = \dfrac{F_{2(n+1)-1}}{L_{2(n+1)-1}}$. $\square$

References

[1] Jaroslav Seibert, ”Solution to Problem B-937 (Proposed by P. Bruckman)”, The Fibonnacci Quarterly, 40 (2) (2003): 87–88.

Problem B-952

B-952 ***Proposed by the Scott H. Brown, Auburn University, Montgomery, AL (February 2003)***

Show that

$$10F_{10n-5} + 12F_{10n-10} + F_{10n-15} = 25F_{2n}^5 + 25F_{2n}^2 + 5F_{2n}$$

for all integers $n \geq 2$.

Solution by Sergio Falcón and Angel Plaza (jointly), Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

Let us call $A = 10F_{10n-5}+12F_{10n-10}+F_{10n-15}$ and $B = 25F_{2n}^5+25F_{2n}^2+5F_{2n}$ We will use the symmetric relations:

$$F_{r+k} = L_k F_r + (-1)^{k+1} F_{r-k} \tag{1}$$

Taking $r = 10n - 10$, and $k = 5$ in (1):

$$F_{10n-5} = L_5 F_{10n-10} + F_{10n-15} = 11F_{10n-10} + F_{10n-15} \tag{2}$$

And substituting in the first part of the identity to be proved, we obtain:
$A = 110F_{10n-10}+10F_{10n-15}+12F_{10n-10}+F_{10n-15} = 122F_{10n-10}+11F_{10n-15} = 123F_{10n-10} + 11F_{10n-15} - F_{10n-10} = L_{10}F_{10n-10} + L_5F_{10n-15} - F_{10n-10}$

where, using (1) with respective values $\{k = 10, r = 10n - 10\}$, and $\{k = 5, r = 10n - 15\}$:
$A = (F_{10n} + F_{10n-20}) + (F_{10n-10} - F_{10n-20}) - F_{10n-10} = F_{10n}$

On the other hand, $F_{2n} = \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n})$, so

$$\begin{aligned}
5F_{2n} &= \tfrac{5}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) \\
25F_{2n}^3 &= \tfrac{5}{\sqrt{5}}(\alpha^{6n} - 3\alpha^{4n}\beta^{2n} + 3\alpha^{2n}\beta^{4n} - \beta^{6n}) = \\
&= \tfrac{5}{\sqrt{5}}(\alpha^{6n} - 3\alpha^{2n} + 3\beta^{2n} - \beta^{6n}) \\
25F_{2n}^5 &= \tfrac{1}{\sqrt{5}}(\alpha^{10n} - 5\alpha^{8n}\beta^{2n} + 10\alpha^{6n}\beta^{4n} - 10\alpha^{4n}\beta^{6n} + 5\alpha^{2n}\beta^{8n} - \beta^{10n}) = \\
&= \tfrac{1}{\sqrt{5}}(\alpha^{10n} - 5\alpha^{6n} + 10\alpha^{2n} - 10\beta^{2n} + 5\beta^{6n} - \beta^{10n})
\end{aligned}$$

where, we have used that $\alpha\beta = -1$. Now, summing up $B = 25F_{2n}^5 + 25F_{2n}^2 + 5F_{2n}$, we also get $B = F_{10n}$. $\square$

Problem B-953

B-953 ***Proposed by Harvey J. Hindin, Huntington Station, NY (February 2003)***

Show that

$$F_n^4 + F_{n+1}^4 + F_{n+2}^4$$

is never a perfect square. Similarly, show that

$$(qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

is never a perfect square, where W_n is defined for all integers n by $W_n = pW_{n-1} - qW_{n-2}$ and where $W_0 = a$ and $W_1 = b$.

Solution by Sergio Falcón and Angel Plaza (jointly), Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

Since $F_{n+2} = F_{n+1} + F_n$, if $A = F_n^4 + F_{n+1}^4 + F_{n+2}^4$, then

$$A = 2\left(F_{n+1}^4 + 2F_{n+1}^3 F_n + 3F_{n+1}^2 F_n^2 + 2F_{n+1} F_n^3 + F_n^4\right) = 2B$$

So if we suppose that A is even, then B is also even.

Now, since any two consecutive terms of the Fibonacci sequence are either both even (case (a)) or one even and the other one odd (case (b)), we have two possibilities:

(a) In this case, $F_{n+1}^4 + F_n^4 + 3F_{n+1}^2 F_n^2$ is an odd number, so B is odd.

(b) Now, if, for example, F_n is even, F_{n+1} is odd, then only F_{n+1}^4 is odd and the rest of the terms into the parenthesis are even, so again B is odd.

In any case, we get a contradiction. $\square$

The reasoning for sequence W_n is similar. Let us call

$$A = (qW_n)^4 + (pW_{n+1})^4 + (W_{n+2})^4$$

Since $W_n = pW_{n-1} - qW_{n-2}$, then $W_{n+2} = pW_{n+1} - qW_n$ so

$$A = (qW_n)^4 + (pW_{n+1})^4 + (pW_{n+1} - qW_n)^4$$

$$A = 2\left[(pW_{n+1})^4 - 2(pW_{n+1})^3(qW_n) + 3(pW_{n+1})^2(qW_n)^2 - 2(pW_{n+1})(qW_n)^3 + (qW_n)^4\right]$$

So A is even.

If $B = \left[(pW_{n+1})^4 - 2(pW_{n+1})^3(qW_n) + 3(pW_{n+1})^2(qW_n)^2 - 2(pW_{n+1})(qW_n)^3 + (qW_n)^4\right]$, then we can distinguish some cases:

(a) If pW_{n+1} and qW_n are odd, then B is odd, because B comprises one odd term $3(pW_{n+1})^2(qW_n)^2$, and the rest of the terms are even.

(b) If pW_{n+1} is odd and qW_n is even, then B is also odd.

(c) Finally, if both pW_{n+1} and qW_n are even, then we have:

$$A = 2{\cdot}16\left[(p'W_{n+1})^4 - 2(p'W_{n+1})^3(q'W_n) + 3(p'W_{n+1})^2(q'W_n)^2 - 2(p'W_{n+1})(q'W_n)^3 + (q'W_n)^4\right]$$

that is $A = 2 \cdot 16B'$. From which B' must be even.

Following this reasoning numbers B, B', etc. are each time lower until we get a contradiction.

$\square$

Problem B-954

B-955 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(February 2003)

Let n be a nonnegative integer. Show that

$$\sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)} = L_{2\lfloor \frac{n}{2}\rfloor+1} + \sqrt{5}F_{2\lceil \frac{n}{2}\rceil}$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor- and ceiling-function, respectively.

Solution by Sergio Falcón and Angel Plaza (jointly), Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

Let us call $A = \sqrt{(\sqrt{5}+2)(\sqrt{5}F_{2n+1}-2)}$, and $B = L_{2\lfloor \frac{n}{2}\rfloor+1} + \sqrt{5}F_{2\lceil \frac{n}{2}\rceil}$.
If $n = 2k$ for some integer k, then $B = L_{2k+1} + \sqrt{5}F_{2k}$, so

$$\begin{aligned} B &= \alpha^{2k+1} + \beta^{2k+1} + \alpha^{2k} - \beta^{2k} = \alpha^{2k}(\alpha+1) + \beta^{2k}(\beta-1) = \\ &= \tfrac{3+\sqrt{5}}{2}\alpha^{2k} - \tfrac{1+\sqrt{5}}{2}\beta^{2k} \end{aligned}$$

So,

$$\begin{aligned} B^2 &= \frac{7+3\sqrt{5}}{2}\,\alpha^{4k} + \frac{3+\sqrt{5}}{2}\,\beta^{4k} - 2(2+\sqrt{5}) = \\ &= \alpha^{4k+1}\,(2+\sqrt{5}) - \beta^{4k+1}\,(2+\sqrt{5}) - 2(2+\sqrt{5}) = \\ &= (2+\sqrt{5})\big(\alpha^{4k+1} - \beta^{4k+1} - 2\big) = (2+\sqrt{5})\big(\sqrt{5}F_{4k+1} - 2\big) = A^2 \end{aligned}$$

Now, if $n = 2k+1$ for some integer k, then

$$\begin{aligned} B &= L_{2k+1} + \sqrt{5}F_{2k+2} = \alpha^{2k}(\alpha+1) + \beta^{2k+1}(1-\beta) = \\ &= \tfrac{3+\sqrt{5}}{2}\alpha^{2k+1} + \tfrac{1+\sqrt{5}}{2}\beta^{2k+1} \end{aligned}$$

So,

$$\begin{aligned} B^2 &= \frac{14+6\sqrt{5}}{4}\,\alpha^{4k+2} + \frac{6+2\sqrt{5}}{4}\,\beta^{4k+2} - (4+2\sqrt{5}) = \\ &= \frac{7+3\sqrt{5}}{2}\,\alpha^{4k+3}\,\frac{2}{1+\sqrt{5}} + \frac{3+\sqrt{5}}{2}\,\beta^{4k+3}\,\frac{2}{1-\sqrt{5}} - (4+2\sqrt{5}) = \\ &= (2+\sqrt{5})\big(\alpha^{4k+3} - \beta^{4k+3}\big) - 2(2+\sqrt{5}) = \\ &= (2+\sqrt{5})\big(\sqrt{5}F_{4k+3} - 2\big) = A^2 \end{aligned}$$

$\square$

Problem B-955

B-955 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 41, no.1, February 2003)***

Prove that

$$1 < \frac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \frac{1}{\sqrt{1+F_{2n+1}^2}} + \frac{1}{\sqrt{1+F_{2n+2}^2}} < \frac{3}{2}$$

for all integer $n \geq 0$.

Solution by Angel Plaza and Sergio Falcón (jointly), Department of Mathematics, University of Las Palmas de Gran Canaria, Spain

Proof: Let us call $A_n = \dfrac{F_{2n}}{\sqrt{1+F_{2n}^2}} + \dfrac{1}{\sqrt{1+F_{2n+1}^2}} + \dfrac{1}{\sqrt{1+F_{2n+2}^2}}$.

For the left inequality, consider: $A_n > \dfrac{F_{2n}}{1+F_{2n}} + \dfrac{1}{1+F_{2n+1}} + \dfrac{1}{1+F_{2n+2}} > 1$, where for the last inequality some elemental algebra must be applied.

For the right inequality, since functions $\dfrac{1}{\sqrt{1+x^2}}$, and $\dfrac{x}{\sqrt{1+x^2}} + \dfrac{2}{\sqrt{1+x^2}}$, are decreasing functions in their arguments, we get that $A_n \leq \dfrac{F_2}{\sqrt{1+F_2^2}} + \dfrac{1}{\sqrt{1+F_3^2}} + \dfrac{1}{\sqrt{1+F_4^2}} = 1.4705491 < \dfrac{3}{2}$ □