881. Proposed by Kim McInturff, Santa Barbara, California.

Prove that for every positive integer n

$$\sum_{k=0}^{n-1}\binom{2k}{k}=\sum_{i=1}^{n}\frac{2\sin\frac{2i\pi}{3}}{\sqrt{3}}\binom{2n}{n+i}.$$

Solution: *(by José M. Pacheco and Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)*

First, notice that the coefficient $\frac{2}{\sqrt{3}}\sin\left(\frac{2i\pi}{3}\right)$ takes the values $+1,-1,0,+1,-1,0,\ldots$, and thus the right-hand side (RHS) may be written as $\displaystyle\sum_{i=1}^{n} i_{[3]}\binom{2n}{n+i}$, where $i_{[3]}$ represents, for $i=1,2,\ldots$, the successive values of the sequence $+1,-1,0,+1,-1,0,\ldots$

The equation may be proved by induction. For the first values of n we have:

- $n=1$: $\binom{0}{0}=+\binom{2}{2}=1$
- $n=2$: $\binom{0}{0}+\binom{2}{1}=+\binom{4}{2+1}-\binom{4}{2+2}=3$
- $n=3$: $\binom{0}{0}+\binom{2}{1}+\binom{4}{2}=+\binom{6}{3+1}-\binom{6}{3+2}+0\binom{6}{3+3}=9$

Let us suppose the expresion is true for n. Now, for $n+1$ it is:

$$(LHS) = \sum_{k=0}^{n}\binom{2k}{k}=\binom{2n}{n}+\sum_{k=0}^{n-1}\binom{2k}{k} \quad (1)$$

$$(RHS) = \sum_{i=1}^{n+1} i_{[3]}\binom{2n+2}{n+1+i} \quad (2)$$

By applying twice the Pascal's rule, $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, then for every integer m, $\binom{2n+2}{m}=\binom{2n}{m-2}+2\binom{2n}{m-1}+\binom{2n}{m}$, and using this into Eq. (2) we have:

$$
\begin{aligned}
(RHS) \quad = \quad & \binom{2n}{n} + \overbrace{2\binom{2n}{n+1}} + \underbrace{\binom{2n}{n+2}} \\
& \overbrace{-\binom{2n}{n+1}} \underbrace{-2\binom{2n}{n+2}} - \cancel{\binom{2n}{n+3}} + 0 + \\
& + \cancel{\binom{2n}{n+3}} + 2\binom{2n}{n+4} + \binom{2n}{n+5} - \dots \\
& +(n+1)_{[3]} \left[\binom{2n}{2n} + \binom{2n}{2n+1} + \binom{2n}{2n+2} \right] = \\
= \quad & \binom{2n}{n} + \binom{2n}{n+1} - \binom{2n}{n+2} + \binom{2n}{n+4} - \binom{2n}{n+5} + \\
& \dots + (n)_{[3]} \binom{2n}{2n} + c_{n+1} \binom{2n}{2n+1} + c_{n+2} \binom{2n}{2n+2}
\end{aligned}
$$

It is easy to see that the coefficient of the term $\binom{2n}{2n}$ is always $(n)_{[3]}$. On the other hand, the coefficients of the last two terms, noted by c_{n+1} and c_{n+2}, depend on the value of $(n+1)_{[3]}$, but these two terms are zero because the lower index is greater than the upper one.

Therefore, (RHS) is the sum of $\displaystyle \binom{2n}{n} + \sum_{i=1}^{n} i_{[3]} \binom{2n}{n+i}$, and by the induction hypothesis $\displaystyle (RHS) = \binom{2n}{n} + \sum_{k=0}^{n-1} \binom{2k}{k} = \sum_{k=0}^{n} \binom{2k}{k} =$ (LHS) and the proof is done. $\square$