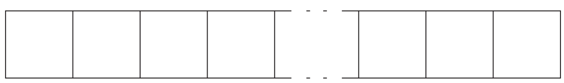
Problem No. 1811. *(Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.)*

Given a connected graph G with vertices $v_1, v_2, \ldots, v_n$, let $d_{i,j}$ denote the distance from v_i to vj. (That is, $d_{i,j}$ is the minimal number of edges that must be traversed in traveling from v_i to v_j.) The Wiener index $W(G)$ of G is defined by

$$W(G) = \sum_{1 \le i < j \le n} d_{i,j}.$$

a. Find tghe Wiener index for the grid-like graph

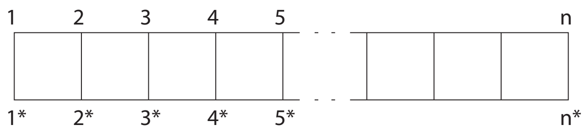


on $2n$vertices.
b. Find ghe Wiener index for the comb-like graph

on $2n$vertices.

Solution by *José M. Pacheco and Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain*

a. Let us number the nodes of the grid-like graph as in the figure



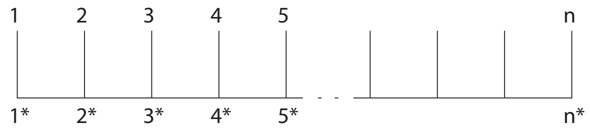
in which the following order $1 < 1^* < 2 < 2^* \cdots (n-1) < (n-1)^* < n < n^*$ for the nodes is considered.

Then, the Wiener index for the grid-like graph is given by the expression

$$\begin{aligned} W(G) &= \sum_{j=1}^{n} \left(\sum_{k=1}^{j-1} 2(j-k) + \sum_{k=1}^{j} (j-k+1) + \sum_{k=1}^{j-1} (j-k+1) \right) \\ &= \sum_{j=1}^{n} (2j^2 - 1) = 2\frac{n(n+1)(2n+1)}{6} - n \\ &= \frac{2n^3 + 3n^2 - 2n}{3} \end{aligned}$$

The reason for the expresion in brackets above comes from the fact that for each j between 1 and n, and for each k between 1 and $j-1$, $d_{k,j} = j-k$. Also for the nodes with astherisc, we have $d_{k^*,j^*} = j-k$. Now, the distance between node k and node j^* is $j-k+1$, while the distance between node k^* and node j is also $j-k+1$, but the range for k is now between 1 and $j-1$ because $j < j^*$.

b. The argument for the comb-like graph as similar as before, considering the figure



in which the same order for the nodes.

Then, the Wiener index for the comb-like graph is given by

$$
\begin{aligned}
W(G) &= \sum_{j=1}^{n}\left(\sum_{k=1}^{j-1}(j-k+2) + \sum_{k=1}^{j-1}(j-k) + \sum_{k=1}^{j}(j-k+1) + \sum_{k=1}^{j-1}(j-k+1)\right) \\
&= \sum_{j=1}^{n}(2j^2+2j-3) = 2\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} - 3n \\
&= \frac{2n^3+9n^2-4n}{3}
\end{aligned}
$$

The only difference with the grid-like graph is the distance between two nodes of the top of the comb-like graph. For this case, $d_{k,j} = j-k+2$ for each j between 1 and n, and for each k between 1 and $j-1$. $\square$