# Solution to MM Problem # 1824

Ángel Plaza and José M. Pacheco
Department of Mathematics, Universidad de Las Palmas de Gran Canaria
35017–Las Palmas G.C. SPAIN,

**Problem# 1824** *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Let $f$ be a continuous real-valued function defined on $[0,1]$ and satisfying

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that there exists a real number $c$, $0 < c < 1$, such that

$$cf(c) = \int_0^c xf(x)dx.$$

**Solution.** Condition

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx$$

implies, by considering the Fourier series of function $f$ in cosines, that both integrals are zero. The solution to the problem is therefore an immediate consequence of the following lemma:

**Lemma.** Let $g$ be a continuous function on $[0,1]$ such that $\int_0^1 g(x)dx = 0$. Then there exists some $a = a_g \in (0,1)$ for which such that $g(a) = \int_0^a g(u)du$.

**Proof:** If $g(x) \equiv 0$ in $[0,1]$, the result is obviously true: Any $a \in (0,1)$ will work. For any nonidentically zero continuous function $g(x)$, let us consider the first subinterval $[x_0, x_1] \subset [0,1]$ where $g$ presents constant sign –without loss of generality we can assume it to be positive– and has no zeroes in its interior.

Now, Let $h(x) = \int_0^x g(u)du$. It is clear that $h(x_0) = 0$ because $g(x) \equiv 0$ in the interval $[0, x_0]$, so $h(x_0) = \int_0^{x_0} g(u)du = \int_0^{x_0} 0du = 0$. Moreover, because $g(x)$ has constant sign in $[x_0, x_1]$, it holds that $h(x)$ is an strictly increasing function on $[x_0, x_1]$, so $h(x_1) \neq 0$. Condition $g(x_1) = 0$ implies that from some $\xi \in (x_0, x_1)$ onwards, $g(x)$ decreases. Therefore, there must exist some $a = a_g \in (x_0, x_1)$ where the decreasing $g(x)$ and the increasing $h(x)$ take the same value. $\square$

*E-mail address*: `aplaza@dmat.ulpgc.es`