Solution to MM Problem # 1827

Ángel Plaza and José M. Pacheco
Department of Mathematics, Universidad de Las Palmas de Gran Canaria
35017–Las Palmas G.C. SPAIN,

Problem# 1827 *Proposed by Christopher Hilliar, Texas A & M University, College Station, TX.* Let A be an $n \times n$ matrix with integer entries and such that each column of A is a permutation of the first column. Prove that if the entries in the first column do not sum to 0, then this sum divides $\det(A)$.

Solution. It is a direct consequence of two basic properties of determinants.

(1) If a row (or column) is changed by adding to or subtracting from its elements any linear combination of the corresponding elements of any other row (or column) the determinant remains unaltered.

(2) If the elements in any row (or column) have a common factor α then the determinant equals the determinant of the corresponding matrix in which $\alpha = 1$.

If we change the first row of matrix A by the sum of all the rows, then all the elements of the new row are equal to the sum or the entries in the first column of matrix A, because the sum of the elements of any permutation independent of the permutation itself. Now, by the second property the result follows. More formally:

Let $(a_1, a_2, \dots, a_n)$ be the entries in the first column of matrix A, and let $\sigma_j \in S_n$, $j = 1, 2, \dots, n-1$ be those permutations of n elements such that $\sigma_j(a_1), \sigma_j(a_2), \dots, \sigma_j(a_n)$ is the j-th column. Then one has:

$$
\begin{aligned}
|A| &= \begin{vmatrix} a_1 & \sigma_1(a_1) & \cdots & \sigma_{n-1}(a_1) \\ a_2 & \sigma_1(a_2) & \cdots & \sigma_{n-1}(a_2) \\ \vdots & \vdots & \ddots & \vdots \\ a_n & \sigma_1(a_n) & \cdots & \sigma_{n-1}(a_n) \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^n a_i & \sum_{i=1}^n \sigma_1(a_i) & \cdots & \sum_{i=1}^n \sigma_{n-1}(a_i) \\ a_2 & \sigma_1(a_2) & \cdots & \sigma_{n-1}(a_2) \\ \vdots & \vdots & \ddots & \vdots \\ a_n & \sigma_1(a_n) & \cdots & \sigma_{n-1}(a_n) \end{vmatrix} \\
&= \sum_{i=1}^n a_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & \sigma_1(a_2) & \cdots & \sigma_{n-1}(a_2) \\ \vdots & \vdots & \ddots & \vdots \\ a_n & \sigma_1(a_n) & \cdots & \sigma_{n-1}(a_n) \end{vmatrix}
\end{aligned}
$$

because for all $j = 1, 2, \dots, n-1$, it holds that $\sum_{i=1}^n a_i = \sum_{i=1}^n \sigma_j(a_i)$. The only restriction is $\sum_{i=1}^n a_i \neq 0$, regardless of the nature of the elements a_i: There is no need to suppose them to be integers.

E-mail address: aplaza@dmat.ulpgc.es