1828. *Proposed by Stephen J. Herschkorn, Department of Statistics, Rutgers University, New Brunswick, NJ.*

Let α_0 be the smallest value of α for which there exists a positive constant C such that

$$\prod_{k=1}^{n}\frac{2k}{2k-1}\leq Cn^{\alpha}$$

for all positive integer n.

a. Find the value of α_0.

b. Prove that the sequence

$$\left\{\frac{1}{n^{\alpha_0}}\prod_{k=1}^{n}\frac{2k}{2k-1}\right\}_{n=1}^{\infty}$$

is decreasing and find its limit.

SOLUTION: *By Santiago de Luxán (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain*

a)

$$A=\prod_{k=1}^{n}\frac{2k}{2k-1}=\frac{(2n)!!}{(2n-1)!!}\leq Cn^{\alpha}$$

Since the inequality must be satisfied for all n ∈ ℕ, if it is true for the maximum value of A, it will happen the same in all cases. Therefore:

$$\frac{(2n)!!}{(2n-1)!!}=\frac{(2n)!!}{\frac{(2n-1)!}{(2n-2)!!}}=\frac{[(2n)!]^2}{2n(2n-1)!}=\frac{2^{2n}[n!]^2}{(2n)!}$$

When n tends to infinite, if we use Stirling at the numerator and the denominator:

$$\frac{2^{2n}[n!]^2}{(2n)!}\sim\frac{2^{2n}n^{n}e^{-2n}2\pi n}{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}=\sqrt{\pi}\frac{n}{\sqrt{n}}=\sqrt{\pi}n^{\frac{1}{2}}$$

which implies that $\alpha_0=\frac{1}{2}$.

b)

If the sequence is decreasing, each term must be smaller than the previous one. Hence:

$$\frac{a_{n+1}}{a_n}\leq 1$$

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{\sqrt{n+1}}\prod_{k=1}^{n+1}\frac{2k}{2k-1}}{\frac{1}{\sqrt{n}}\prod_{k=1}^{n}\frac{2k}{2k-1}}=\sqrt{\frac{n}{n+1}}\frac{\prod_{k=1}^{n}\frac{2k}{2k-1}}{\prod_{k=1}^{n}\frac{2k}{2k-1}}\frac{2(n+1)}{2(n+1)-1}=\sqrt{\frac{n}{n+1}}\frac{2n+2}{2n+1}=$$

$$=2\frac{\sqrt{n(n+1)}}{2n+1}=\sqrt{\frac{4n^2+4n}{4n^2+4n+1}}\leq 1\, for\ all\, n\in\ \mathbb{N}$$

Now we find the limit of the sequence:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\prod_{k=1}^{n}\frac{2k}{2k-1}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{(2n)!!}{(2n-1)!!}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{2^{2n}[n!]^2}{(2n)!}$$

Again, we use Stirling:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\frac{2^{2n}n^{2n}e^{-2n}2\pi n}{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}=\sqrt{\pi}\lim_{n\to\infty}\frac{n}{\sqrt{n}\sqrt{n}}=\sqrt{\pi}$$

Notice that since $\{a_n\}$ is decreasing, then

$$a_1=2\geq a_n\geq\sqrt{\pi}$$

$$2\geq\frac{\prod_{k=1}^{n}\frac{2k}{2k-1}}{\sqrt{n}}\geq\sqrt{\pi}$$

$$\sqrt{\pi}\sqrt{n}\leq\prod_{k=1}^{n}\frac{2k}{2k-1}\leq 2\sqrt{n}\ for\ all\ n\in\ \mathbb{N},$$

we deduce that the C from a) must be ≥ 2.