2005. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Let $f : [0, \infty)(0, \infty)$ be an increasing, differentiable function with continuous derivative, and let k be a nonnegative integer. Prove that

$$\int_0^\infty \frac{x^k}{f(x)} dx$$

converges if and only if

$$\int_0^\infty \frac{x^k}{f(x) + f'(x)} dx$$

converges.

Solution: *(by José Miguel Pacheco and Ángel Plaza, ULPGC, 35017-Las Palmas G.C., Spain, e-mail: pacheco@dma.ulpgc.es, aplaza@dmat.ulpgc.es)*

The $\Rightarrow$ part is immediate: If the first integral converges, the second one also converges:
The condition of growing f is expressed as $f'(x) > 0$ for every $x > 0$, therefore the denominator in the second integrand is larger than that in the first one, and the theorem on dominated convergence immediately applies.

The $\Leftarrow$ part is an exercise on orders of infinity. The first integral is convergent for $f(x) \approx x^{k+1+\epsilon}$, and the second one, for $f(x) + f'(x) \approx x^{k+1+\epsilon}$. But the order of infinity of $f'(x)$ is always one unit less than the order of $f(x)$, so the infinity order of $f(x) + f'(x)$ is the same as that of $f(x)$. Therefore, if the second integral converges, so does the first one.

Appendix

The order of infinity of $f'(x)$ is one unit less than that of $f(x)$.
It is a consequence of the formal Taylor development up to the first

order: From $f(x+a) = f(x) + af'(x+\theta)$ it follows that:

$$\begin{aligned} f'(x+\theta) &= \frac{f(x+a)-f(x)}{a} = (x+a)\frac{f(x+a)-f(x)}{(x+a)a} \\ &= \left(1+\frac{x}{a}\right)\frac{f(x+a)-f(x)}{x+a}. \end{aligned}$$

In this last expression, let first $a \to \infty$, and then $x \to \infty$, to obtain $f'(x) \approx \dfrac{f(x)}{x}$.

Reference

DuBois-Reymond, Paul (1875) *Uber asymptotische Werthe, infinitare Approximationen und infinitare Auflosung von Gleichungen*, Mathematische Annalen, **8**, 363-414.

The above computation is in pages 368-369.