Problem 1769 (Proposed by Michel Bataille, Rouen, France)

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For positive integer n, let

$$P_n(x,y) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} x^{n-k}(x+y)^k.$$

Find a closed form expression for the coefficient of $x^i y^j$ when P_n is expanded.

Solution:
Since $(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$, then

$$\begin{aligned} P_n(x,y) &= \sum_{k=0}^{n} \binom{2n+1}{2k+1} x^{n-k} \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j \\ &= \sum_{k=j}^{n} \binom{2n+1}{2k+1} \binom{k}{j} x^{n-j} y^j \end{aligned}$$

Note that all the terms when P_n is expanded are of the form $x^{n-j}y^j$. We will prove, by using the *snake oil method* [1] that

$$\sum_{k=j}^{n} \binom{2n+1}{2k+1} \binom{k}{j} = 4^{n-j} \binom{2n-j}{j} \tag{1}$$

In order to show previous identity we will obtain the generating funcion of both sides of Eq. (1). We will use the following identity (see [1, Eq. (4.3.1), page 120]);

$$\sum_{r\geq 0} \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}} \qquad (k \geq 0)$$

$$F(x) = \sum_{n\geq 0} \sum_{k=j}^{n} \binom{2n+1}{2k+1} \binom{k}{j} x^n$$

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$$
\begin{aligned}
&= \sum_{k\geq 0} \frac{\binom{k}{j}}{\sqrt{x}} \sum_{n\geq 0} \binom{2n+1}{2k+1} (\sqrt{x})^{2n+1} = \\
&= \frac{1}{\sqrt{x}} \sum_{k\geq 0} \binom{k}{j} \frac{(\sqrt{x})^{2k+1}}{(1-\sqrt{x})^{2k+2}} = \\
&= \sum_{k\geq 0} \binom{k}{j} \frac{x^k}{(1-\sqrt{x})^{2k+2}} = \\
&= \frac{1}{(1-\sqrt{x})^2} \sum_{k\geq 0} \binom{k}{j} \left(\frac{x}{(1-\sqrt{x})^2} \right)^k \\
&= \frac{1}{(1-\sqrt{x})^2} \cdot \frac{\left(\frac{x}{(1-\sqrt{x})^2} \right)^j}{\left(1 - \frac{x}{(1-\sqrt{x})^2} \right)^{j+1}} \\
&= \frac{x^j}{(1-2\sqrt{x})^{j+1}}
\end{aligned}
$$

Now we will obtain the generating function for the right-hand side of Eq. (1):

$$
\begin{aligned}
G(x) &= \sum_{n\geq 0} 4^{n-j} \binom{2n-j}{j} x^n \\
&= \sum_{n\geq 0} \left(\frac{\sqrt{x}}{2} \right)^j \binom{2n-j}{j} \left(2\sqrt{x}\right)^{2n-j} \\
&= \left(\frac{\sqrt{x}}{2} \right)^j \cdot \frac{(2\sqrt{x})^j}{(1-2\sqrt{x})^{j+1}} \\
&= \frac{x^j}{(1-2\sqrt{x})^{j+1}}
\end{aligned}
$$

and Identity (1) is done. □

References

[1] Herbert S. Wilf, Generatingfunctionology, Academic Press, Inc., Second ed. 1994.

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