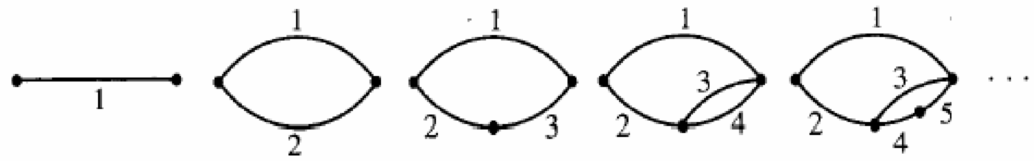
PROBLEMAS DE REVISTAS INTERNACIONALES PROPUESTOS EN EL CURSO 2005—2006

Mathematics Magazine, Vol. 78, No. 3, Jun 2005 Deadline: November 1, 2005

1721. *Proposed by Donald Knuth, Stanford University, Stanford, CA.*

The *Fibonacci graphs*



are defined by successively replacing the edge with maximum label n by two edges n and $n+1$, in series if n is even, and in parallel if n is odd. Prove that the Fibonacci graph with n edges has exactly F_{n+1} spanning trees, where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Show also that these spanning trees can be listed in such a way that some edge k is replaced by $k \pm 1$ as we pass from one tree to the next. For example, for $n = 5$ the eight spanning trees can be listed as 125, 124, 134, 135, 145, 245, 235, 234.

1722. *Proposed by Emeric Deutsch, Polytechnic University, Broooklyn, NY.*

Let k and n be positive integers with $k \leq n$. Find the number of permutations of $\{1, 2, \ldots, n\}$ in which $1, 2, \ldots, k$ appears as a subsequence but $1, 2, \ldots, k, k+1$ does not.

1723. *Proposed by Herb Bailey, Rose Hulman Institute of Technology, Terre Haute, IN.*

Let I be the incenter of triangle ABC with BC tangent to the incircle at D. Let E be the intersection of the extension of $\overline{ID}$ with the circle through B, I, and C. Prove that

$$\overline{DE} = \frac{T}{s-a},$$

were T and s are, respectively, the area and semiperimeter of triangle ABC, and $a = \overline{BC}$.

1724. *Proposed by Mihály Bencze, Săcele-Négyfalu, Romania.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{n}\sum_{k=1}^{k} x_k - \left(\prod_{k=1}^{n} x_k\right)^{1/n} \leq \frac{1}{n} \sum_{1\leq j<k\leq n} (\sqrt{x_j} - \sqrt{x_k})^2.$$

1725. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{E}$ be the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, where a and b are positive integers. Find the number of parallelograms with vertices at integer lattice points and sides tangent to $\mathcal{E}$ at their midpoints.

Math Horizons, Vol. 13, No. 1, Sep 2005   Deadline: November 10, 2005

S98. *Proposed by Michael Lanstrum, Cuyahoga Community College, Parma.* Solve the equation $(x^2 - 5x + 5)^{x^2-5x} = 1$.

S99. *Proposed by Mircea Ghita, Flushing.* Let n be a positive integer. Solve the equation

$$\sum_{k=1}^{n} \cos^k kx = \frac{n(n+1)}{2}.$$

S100. *Proposed by Linda Yu, Montreal, Canada.* Lily chooses seven nonnegative numbers with sum 1, and Lala places them around a circle. Lily's score is the highest value among the seven products of adjacent numbers. What is the highest score she can get, with Lala trying to make it as low as possible?

Problem 195. *Proposed by Juan-Bosco Romero Marquez, Universidad de Valladolid, Spain.* Find all solutions (x,y) in positive integers to the equation $(x + y)^x - x^{x+y} = 2$.

Problem 196. *Proposed by Frank Flanigan, San Jose State University, San Jose.* The power series $a_0 + a_1x + a_2x^2 + \cdots$ has nonnegative coefficients not all of which are zero, and is convergent for all real numbers x. Define $b_n = a_n + a_{n-2} + a_{n-4} + \cdots$ where the last term is a_0 or a_1 dependent on the parity of n. Is the power series $b_0 + b_1x + b_2x^2 + \cdots$ also convergent for all real numbers x?

College Mathematics Journal, Vol. 36, No. 4, Sep 2005   Deadline: December 15, 2005

PROBLEMS

796 (corrected). *Proposed by Andrew Cusamano, Great Neck, NY.*

Let the Fibonacci sequence be defined as $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$. Let $\alpha = \frac{1+\sqrt{5}}{2}$.

(a) Show that

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{f_k f_{k+1}}$$

converges to α.

(b) Let $N \in \mathbb{Z}^+$. Express each a_k in terms of Fibonacci numbers so that the series

$$1 + \sum_{k=1}^{\infty} \frac{a_k}{(f_k f_{k+1})^N}$$

converges to α^N.

806. *Proposed by Kenneth Fogarty, East Branch, NY 13756.*

Given $\triangle ABC$, construct a sequence of derived triangles in the following manner:

(i) Take $\triangle ABC$ as the first triangle in the sequence and denote it as $\triangle A_0B_0C_0$.

(ii) Given $\triangle A_kB_kC_k$ in the sequence, let the circle inscribed in $\triangle A_kB_kC_k$ intersect $\overline{A_kB_k}$, $\overline{B_kC_k}$, and $\overline{C_kA_k}$ at points P_k, Q_k, and R_k, respectively. If the lengths of $\overline{A_kP_k}$, $\overline{B_kQ_k}$, and $\overline{C_kR_k}$ can be used to form a triangle, then take $\triangle A_{k+1}B_{k+1}C_{k+1}$ to be a triangle for which the lengths of the sides are $\overline{A_{k+1}B_{k+1}} = 2\overline{C_kR_k}$, $\overline{B_{k+1}C_{k+1}} = 2\overline{A_kP_k}$, and $\overline{C_{k+1}A_{k+1}} = 2\overline{B_kQ_k}$. If the lengths of $\overline{A_kP_k}$, $\overline{B_kQ_k}$, and $\overline{C_kR_k}$ cannot be used to form a triangle, then the sequence terminates at $\triangle A_kB_kC_k$.

Consider the following questions:

(a) For a natural number n, is there a triangle for which the sequence of derived triangles has length n?

(b) For which triangles does the sequence of derived triangles never terminate?

807. *Proposed by Angelo S. DiDomenico, Milford, MA.*

Let a, b, and c be positive integers such that $\gcd(a, c) = \gcd(b, d) = 1$ and b, d have opposite parity.

(a) Find necessary and sufficient conditions under which $bc = ab + ad + cd$.

(b) In part (a), let $x = bc - ab$, $y = bc - cd$, and $z = bc - ad$. Show that (x, y, z) generates all primitive Pythagorean triples.

808. *Proposed by Yongge Tian, University of Alberta, Edmonton, Canada.*

Let

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

be a symmetric positive definite matrix. Show that

$$\operatorname{rank}\left([X_1^T, X_2^T]V^{-1} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - X_1^T V_{11}^{-1} X_1 \right) = \operatorname{rank}(X_2 - V_{21} V_{11}^{-1} X_1),$$

where $(.)^T$ denotes transpose.

809. *Proposed by Ovidui Furdui, Western Michigan University, Kalamazoo, MI.*

Find

$$\lim_{n\to\infty} \left\{ \left[\sum_{k=n+1}^{2n} \left(2\sqrt[2k]{2k} - \sqrt[k]{k} \right) \right] - n \right\}.$$

810. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i\le j\le n} \left(\frac{n^4 + (i^2 + j^2)n^2 + i^2 j^2}{n^4 + i^2 + j^2} \right)^{1/2}.$$

Mathematics Magazine, Vol. 78, No. 4, Oct 2005 Deadline: March 1, 2006

Proposals

To be considered for publication, solutions should be received by March 1, 2006.

1726. *Proposed by Jerry Metzger, University of North Dakota, Grand Forks, ND.*

Let a and j be positive integers with $a \geq 2$. Show that there is a positive integer n such that $a^n \equiv -j \pmod{a^j + 1}$ if and only if $j = a^k$ for some $k \geq 0$.

1727. *Proposed by Jody M. Lockhart and William P. Wardlaw, U. S. Naval Academy, Annapolis, MD.*

Chain addition is a technique used in cryptography to extend a short sequence of digits, called the seed, to a longer sequence of pseudorandom digits. If the seed sequence of digits is $a_1, a_2, \ldots, a_n$, then for positive integer k, $a_{n+k} = a_k + a_{k+1} \pmod{10}$, that is, a_{n+k} is the units digit in the sum $a_k + a_{k+1}$. Suppose that the seed sequence is 3, 9, 6, 4. Prove that the sequence is periodic and find, without the use of calculator or computer, the number of digits in the sequence before the first repetition of 3, 9, 6, 4.

1728. *Proposed by José Luis Díaz-Barrero, Universitat Politènica de Catalunya, Barcelona, Spain.*

Let $A_1A_2\ldots A_{3n}$ be a regular polygon with $3n$ sides, and let P be a point on the shorter arc A_1A_{3n} of its circumcircle. Prove that

$$\left(\sum_{k=1}^{n} PA_{n+k}\right)\sum_{k=1}^{n}\left(\frac{1}{PA_k} + \frac{1}{PA_{2n+k}}\right) \geq 4n^2.$$

1729. *Proposed by Brian T. Gill, Seattle Pacific University, Seattle, WA*

For positive integer k, let c_k denote the product of the first k odd positive integers, and let $c_0 = 1$. Prove that for each nonnegative integer n,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{n!}{k!}2^{n-k}c_k = c_n.$$

1730. *Proposed by Steven Butler, University of California San Diego, La Jolla, CA.*

Let A and B be symmetric, positive semi-definite matrices such that $A + B$ is positive definite, and let $\|\mathbf{y}\|$ denote the usual 2-norm of the vector $\mathbf{y}$. Prove that for all $\mathbf{x} \neq \mathbf{0}$,

$$\left\|(I - A)(I + A)^{-1}(I - B)(I + B)^{-1}\mathbf{x}\right\| < \|\mathbf{x}\|.$$

1718. *Proposed by David Callan, Madison, WI.*

Let k, n be integers with $1 \le k \le n$. Prove the identity

$$\sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n-(k-1)}{k-i} 2^{k-i-1} = \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{n-i}{k}.$$

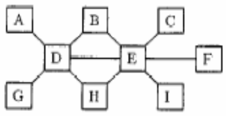
Note. This corrects a typo in the April 2005 appearance of the problem.

Math Horizons, Vol. 13, No. 2, Nov 2005 Deadline: January 10, 2006

Proposals

To be considered for publication, solutions to the following problems should be received by January 10, 2006.

S101. *Proposed by R.G. Watkins, Hemet, CA.* In the diagram below, replace each of the letters by a distinct nonzero digit so that the sum of the digits in three boxes connected by a line is constant. Give an example for each possible value of this magic sum.



S102. *Proposed by Linda Yu, Montreal.* In the US Cyber League, twelve teams played one another 16 times in a season. The numbers of games played among the teams so far were shown in the chart below, with the teams identified by their Roman numerals only. That is, determine the identities of teams X and XII.

#	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
I	0	8	12	10	13	5	8	8	8	10	11	14
II	8	0	11	9	12	6	7	9	7	11	10	13
III	12	11	0	6	5	9	4	12	6	14	7	10
IV	10	9	6	0	7	7	2	10	4	12	5	6
V	13	12	5	7	0	10	5	13	7	15	8	11
VI	5	6	9	7	10	0	5	5	5	7	8	11
VII	8	7	4	2	5	5	0	8	2	10	3	6
VIII	8	9	12	10	13	5	8	0	8	6	11	14
IX	8	7	6	4	7	5	2	8	0	10	5	8
X	10	11	14	12	15	7	10	6	10	0	13	16
XI	11	10	7	5	8	8	3	11	5	13	0	9
XII	14	13	10	6	11	11	6	14	8	16	9	0

The numbers of games played between certain pairs of teams were known:

First Team	Second Team	#
New York Netscapes	Seattle Spacebars	4
Los Angeles Laptops	San Francisco Spams	4
Philadelphia Printers	Miami Modems	5
Philadelphia Printers	Washington Websites	5
Chicago Computers	Washington Websites	6
Denver Desktops	Boston Bitmaps	8
Denver Desktops	Detroit Debuggers	9
New York Netscapes	St. Louis Scanners	11
Chicago Computers	Los Angeles Laptops	12

Which two teams have played all the games between them?

S103. *Proposed by Mircea Ghita, Humanities High School, New York.* For each positive integer n, determine all real values of x in the interval $(0, \pi/4)$ such that

$$\sin^n x + \cos^n x + \tan^n x + \cot^n x + \sec^n x + \csc^n x = 7.$$

Problem 197. *Proposed by U. I. Lydna, Beloretsk, Russia.* D is a point on side BC of triangle ABC such that AD bisects $\angle CAB$. E and F are the respective feet of the perpendicular from D onto CA and AB. H is the point of intersection of BE and CF. G and I are the respective feet of the perpendicular from D to BE and CF. Prove that the quadrilaterals $AFGH$ and $AEIH$ are cyclic (i.e., the vertices lie on a circle).

Problem 198. *Proposed by Serhiy Grabarchuk, Uzhgorod, Ukraine.*

(a) Each of the 9 cells of a 3×3 configuration is either yellow or blue. How many distinct patterns are there, up to rotations, reflections and color reversals?

(b) A 5×5 piece of origami paper is yellow on one side and blue on the other. It is folded into a 3×3 configuration such that each of the 9 cells on one side is either yellow or blue. How many of the distinct patterns in (a) can be obtained by this process?

College Mathematics Journal, Vol. 36, No. 5, Nov 2005 Deadline: February 15, 2006

PROBLEMS

811. *Proposed by William Wardlaw, U.S. Naval Academy, Annapolis, MD.*

Suppose that A is a matrix over the rational numbers with classical adjoint (adjugate) B and characteristic polynomial $f_A(x) = \det(xI - A) = x^5 + 2x^4 + 3x^3 - x^2 + x + 2$. What is the characteristic polynomial $f_B(x)$ of B?

812. *Proposed by Ovidui Furdui, Western Michigan University, Kalamazoo, MI.*

Calculate the limit

$$\lim_{n\to\infty} \left(e^{\sin 1/1+\sin 1/2+\cdots+\sin 1/n+1} - e^{\sin 1/1+\sin 1/2+\cdots+\sin 1/n}\right).$$

813. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN.*

Suppose there are 45 nested equilateral triangles $A_1B_1C_1, A_2B_2C_2, \ldots, A_{45}B_{45}C_{45}$ with sides of length

$$\cos 1^\circ, \cos 3^\circ, \ldots, \cos 89^\circ,$$

respectively. Also, suppose that P_i is any point inside the ith triangle and that P_{i1}, P_{i2}, and P_{i3} are the lengths of the perpendiculars from P_i to the three sides of that triangle. Show that

$$\frac{4\sqrt{3}}{3}\sum_{i=1}^{45}(P_{i1} + P_{i2} + P_{i3}) = \csc 1^\circ.$$

814. *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Suppose that a, b, and c are positive real numbers such that $ab + bc + ca = a + b + c$. Find the minimum value of

$$\frac{a^3}{(b-c)^2 + bc} + \frac{b^3}{(c-a)^2 + ca} + \frac{c^3}{(a-b)^2 + ab}.$$

815. *Proposed by Yongge Tian, University of Alberta, Edmonton, Canada.*

Suppose that A_i and B_i are $m_i \times n_i$ and $m_i \times p_i$ nonzero matrices, respectively, for $i = 1, \ldots, k$. Show that

$$\operatorname{range}(A_1 \otimes A_2 \otimes \cdots \otimes A_k) \subseteq \operatorname{range}(B_1 \otimes B_2 \otimes \cdots \otimes B_k)$$

if and only if

$$\operatorname{range}(A_i) \subseteq \operatorname{range}(B_i), \quad i = 1, \ldots, k,$$

where $\otimes$ denotes the Kronecker product of matrices.