A Mean Value Formula

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Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function with f'' continuous such that $f(0) = f(1)$. Show that there exists $\xi \in (0,1)$ such that

$$\xi^2 f(\xi) = 2\int_0^{\xi} x f(x) dx.$$

Solution: *by Ángel Plaza and Sergio Falcón, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, 35017–Las Palmas G.C. SPAIN, aplaza@dmat.ulpgc.es.*

Let us consider function $h(t) = \displaystyle\int_0^t x^2 f'(x)dx$. Since $h(t) = t^2 f(t) - 2\displaystyle\int_0^t x f(x)dx$, the question is equivalent to the existence of $\xi \in (0,1)$ such that $h(\xi) = 0$.

By hypothesis of the problem $f(0) = f(1)$. If function f is constant in some subinterval $[0,a] \subset [0,1]$, the problem is trivial. In other case, we can suppose without loss of generality that $f(t) \neq f(0)$ for all $t \in (0,1)$. Otherwise we would consider the same question for the interval $[0,a] \subset [0,1]$, being a the first point in $(0,1)$ with $f(a) = f(0)$.

Since $f(t) \neq f(0)$ for all $t \in (0,1)$, let us suppose WLOG that $f(t) > f(0) \geq 0$ for all $t \in (0,1)$. By Rolle's theorem, there is a point $c \in (0,1)$ such that $f'(c) = 0$. We can suppose WLOG that $\operatorname{sign}(f'(t))$ is positive for all $t \in (0,c)$, and also is positive $\operatorname{sign}(t^2 f'(t))$ in the same interval.

Note that $h(0) = 0$, $h(c) > 0$ and $h(1) = \displaystyle\int_0^1 x^2 f'(x)dx < \int_0^1 f'(x)dx = f(1) - f(0) = 0$. Hence, by Bolzano's theorem there is a point $\xi \in (c,1)$ such that $h(\xi) = 0$. Notice that apparently the hypothesis of f'' continuous is superfluous. $\square$