Problem No. 1806. *(Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.)*

The intersection of the ellipsoid $x^2+y^2+\frac{z^2}{c^2}=1$ and the plane $x+y+cz=0$ is an ellipse. For $c>1$, find the value of c for which the area of the ellipse is maximal.

Solution by *José M. Pacheco and Ángel Plaza, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain*

The plane $x+y+cz=0$ intersects the XY plane along its secondary diagonal, and the dihedral angle depends on c. Because the given ellipsoid is a revolution surface around the OZ axis (for $c=1$ it becomes a sphere) the problem can be cast in a simpler form. Indeed, for $c=1$ the problem boils down to the computation of the area of a circle.

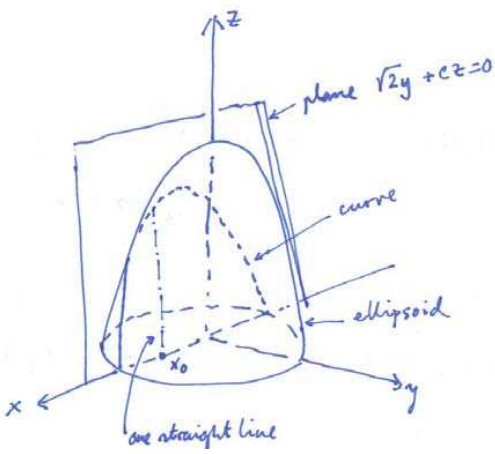

Figure 1: Geometry of the problem after rotating about the $0Z$ axis

To make things easier, let us rotate everything through $\dfrac{\pi}{4}$ radians around the OZ axis, so the plane $x+y+cz=0$ can be changed into $\sqrt{2}y+cz=0$ intersecting the XY plane along the OX axis. The orthogonal vector to this plane is $(0,\sqrt{2},c)$, so all straight lines with base point in the OX axis and having $(0,-c,\sqrt{2})$ as their common direction vector lie on $\sqrt{2}y+cz=0$ at right angles with OX. See Figure 1. The parametric equations of this line

family are:

$$\begin{cases} x &= x_0 \\ y &= -c\,\lambda \\ z &= \sqrt{2}\,\lambda \end{cases}$$

By plugging these values in the ellipsoid equation we obtain $x_0^2 + c^2\lambda^2 + \dfrac{2\lambda^2}{c^2} = 1$, or else $\lambda^2\left(c^2 + \dfrac{2}{c^2}\right) = 1 - x_0^2$, from where $\lambda^2 = \left(c^2 + \dfrac{2}{c^2}\right)^{-1/2}\sqrt{1 - x_0^2}$.

For any $x \in [-1, 1]$, this choice of λ yields a point lying both in the straight line and on the ellipsoid, (*i.e.* in the curve we are looking for), say $P_\lambda(x_0)$: Actually there are two of them, but one is enough for our purpose. The plane area enclosed by the curve is given by

$$A(c) = 4\int_0^1 \operatorname{dist}\left[(x_0, 0, 0), P_\lambda\right] dx_0$$

In this formula we shall use the fact that

$$\operatorname{dist}^2\left[(x_0, 0, 0), P_\lambda\right] = (c^2 + 2)\lambda^2 = (c^2 + 2)\frac{1 - x_0^2}{c^2 + \dfrac{2}{c^2}} = G(c)(1 - x_0^2)$$

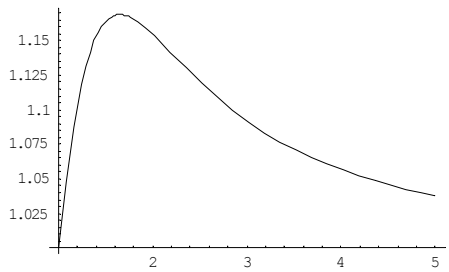

Figure 2: Graph of $g(c)$

Therefore, $A(c) = 4g(c)\displaystyle\int_0^1 (1 - x_0^2)^{1/2} d\,x_0 = k\;g(c)$, where the constant $k = 4\displaystyle\int_0^1 (1 - x_0^2)^{1/2} dx_0$, and $g(c) = \sqrt{G(c)} = \left(\dfrac{c^4 + 2c^2}{c^4 + 2}\right)^{1/2}$.

Thus, $A(c)$ has a maximum if and only if $g(c)$ has one. The graph of $g(c)$ is shown in Figure 2. A "little algebra" yields that $g'(c) = 0$ has the unique real solution $c_{max} = \left(1 + \sqrt{3}\right)^{1/2} > 1$. $\square$