PROBLEM OF THE WEEK

Problem No. 13 (Fall 2010 Series)

Let f be a non–negative, continuous function on the interval $0 \leq x \leq 1$, and suppose that

$$\int_0^x f(t)dt \geq f(x)$$

for all such x. Prove that f vanishes identically.

Solution: by Ángel Plaza ULPGC, Spain

Since f is a non–negative continuous function on the interval $0 \leq x \leq 1$, then $F(x) = \displaystyle\int_0^x f(t)dt$ also is a non–negative continuous function on the interval $0 \leq x \leq 1$.

Therefore,

$$\int_0^x f(t)dt \geq f(x) \Rightarrow \int_0^x \left(\int_0^t f(s)ds \right) dt \geq \int_0^x f(t)dt.$$

But the left-hand side integral in the last inequality, by Cauchy's reult may be written as $\displaystyle\int_0^x \left(\int_0^t f(s)ds \right) dt = \int_0^x (x-t)f(t)dt$, and hence the last inequality becomes

$$\begin{aligned} & \int_0^x (x-t)f(t)dt \geq \int_0^x f(t)dt \\ \int_0^x f(t)dt \quad \geq & \int_0^x (x-t)f(t)dt \geq \int_0^x f(t)dt \end{aligned}$$

which it is true if and only if f vanishes identically. $\square$